ERGODIC THEORETIC PROOF OF EQUIDISTRIBUTION OF HECKE POINTS

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1. Introduction

Let G be a connected non-compact $\mathbb{Q}$-simple real algebraic group defined over $\mathbb{Q}$, that is, the identity component of the group of the real points of a connected $\mathbb{Q}$-simple algebraic group which is $\mathbb{R}$-isotropic. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup of G. As is well known, Γ has finite co-volume in G [BH]. Denote by μ_G the G-invariant Borel probability measure on $\Gamma\backslash G$. Two subgroups Γ_1 and Γ_2 of G are said to be *commensurable* with each other if $\Gamma_1 \cap \Gamma_2$ has a finite index both in Γ_1 and Γ_2. The commensurator group $\mathrm{Comm}(\Gamma)$ of Γ is defined as follows:

$$\mathrm{Comm}(\Gamma) = \{g \in G : \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable with each other}\}.$$

Since Γ is an arithmetic subgroup, $\mathrm{Comm}(\Gamma)$ contains $G(\mathbb{Q})$ and in particular, Γ has an infinite index in $\mathrm{Comm}(\Gamma)$.

For an element $a \in \mathrm{Comm}(\Gamma)$, the Γ-orbit $\Gamma\backslash\Gamma a\Gamma$ in $\Gamma\backslash G$ has finitely many points called *Hecke points* associated with a. We set

$$\deg(a) = \#\Gamma\backslash\Gamma a\Gamma.$$

It is easy to see that $\deg(a) = [\Gamma : \Gamma \cap a^{-1}\Gamma a]$.

In this paper, we are interested in the equidistribution problem of the Hecke points $\Gamma\backslash\Gamma a\Gamma$ as $\deg(a) \to \infty$. Namely, for any continuous function f in $\Gamma\backslash G$ with compact support, any $x \in \Gamma\backslash G$, and for any sequence $a_i \in \mathrm{Comm}(\Gamma)$ with $\deg(a_i) \to \infty$,

$$\text{(1.1)} \qquad \text{Does } \mathrm{T}_{a_i}(f)(x) := \frac{1}{\deg(a_i)} \sum_{\gamma \in \Gamma\backslash\Gamma a_i\Gamma} f(\gamma x) \text{ converge to } \int_{\Gamma\backslash G} f \, d\mu_G?$$

We remark that $\mathrm{T}_{a_i}(f)$ is well defined as a function on $\Gamma\backslash G$. When G is simple, $a_i \in G(\mathbb{Q})$ and Γ is a *congruence* subgroup, this was answered in the affirmative way in [COU], based on the adelic interpretation of $L^2(\Gamma\backslash G)$ and an information on the local harmonic analysis of the p-adic groups $G(\mathbb{Q}_p)$. For smooth functions, the methods in [COU] also give rate of the convergence. Several partial results in this direction were known (see [C], [CU], [Sa], [GM], etc.)

It was pointed out by Burger and Sarnak [BS, "Theorem 5.2"] in 1991 that the equidistribution of Hecke points follows from Ratner's measure classification theorem [Ra] provided a_i's converge to an element not belonging to $\mathrm{Comm}(\Gamma)$. However Burger and Sarnak did not

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give a detailed proof of this claim: this was done by Dani and Margulis [DM2, Corollary 6.2] in 1993, where they deduced the assertion from their ergodic results built up on Ratner's fore-mentioned theorem.

Unlike the method used in [COU], this ergodic theoretic method does not provide a rate information of the equidistribution. However it works for much more general cases, for instance, Γ can be a non-congruence subgroup and a_i's need not be necessarily elements of $G(\mathbb{Q})$.

Our main purpose of this paper is to present an ergodic theoretic proof of the following result:

Theorem 1.2. *Let G be a connected non-compact $\mathbb{Q}$-simple real algebraic group defined over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup of G. Let $\{a_i \in \mathrm{Comm}(\Gamma)\}$ be a sequence such that $\lim_{i\to\infty} \deg(a_i) = \infty$. Then for any bounded continuous function f on $\Gamma\backslash G$ and for any $x \in \Gamma\backslash G$,*

$$\lim_{i\to\infty} \mathrm{T}_{a_i}(f)(x) = \int_{\Gamma\backslash G} f(g)\, d\mu_G(g).$$

2. Limits of H-invariant measures

Let G be a connected real semisimple algebraic group defined over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup of G. Let H be a connected real non-compact semisimple $\mathbb{Q}$-simple subgroup of G. Then $\Gamma \cap H$ is an irreducible (H being $\mathbb{Q}$-simple) Zariski dense lattice in H.

Let $\{g_m \in G\}$ be a sequence such that $g_m^{-1}\Gamma g_m \cap H$ is commensurable with $\Gamma \cap H$. It follows that each $\Gamma\backslash\Gamma g_m H$ is closed [Rag] and there exists the unique H-invariant probability measure, say ν_m, in $\Gamma\backslash G$ supported on $\Gamma\backslash\Gamma g_m H$.

Let Y denote $\Gamma\backslash G$ if $\Gamma\backslash G$ is compact; and otherwise the one point compactification $\Gamma\backslash G \cup \{\infty\}$. The space $\mathcal{P}(Y)$ of the probability measures on Y equipped with the weak*-topology is weak* compact.

Our basic tools for the proof of Theorem 1.2 are the following two propositions: Denote by H_N the unique maximal connected normal subgroup of H with no compact factors.

Proposition 2.1. *Suppose that $g_m H_N g_m^{-1}$ is not contained in any proper parabolic $\mathbb{Q}$-subgroup of G for each m. Then every weak limit of $\{\nu_m : m \in \mathbb{N}\}$ in $\mathcal{P}(Y)$ is supported in $\Gamma\backslash G$.*

This proposition is shown in [EO, Proposition 3.4] based on theorems of Dani and Margulis ([DM1, Theorem 2] and [DM2, Theorem 6.1]).

Proposition 2.2. *Suppose that ν_m weakly converges to a measure ν in $\mathcal{P}(\Gamma\backslash G)$ as $m \to \infty$. Then there exists a closed connected subgroup L of G containing H such that*

1. *ν is an L-invariant measure supported on $\Gamma\backslash\Gamma c_0 L$ for some $c_0 \in G$;*
2. *$\Gamma \cap c_0 L c_0^{-1}$ is a Zariski dense lattice in $c_0 L c_0^{-1}$ and hence in particular $c_0 L c_0^{-1}$ is defined over $\mathbb{Q}$;*
3. *there exist $m_0 \in \mathbb{N}$ and a sequence $\{x_m \in \Gamma g_m H\}$ converging to c_0 as $m \to \infty$ such that $c_0 L c_0^{-1}$ contains the subgroup generated by $\{x_m H x_m^{-1} : m \geq m_0\}$.*

This proposition is deduced from the following theorem of Mozes and Shah:

Theorem 2.3 (MS, Theorem 1.1)**.** *Let $\{u_i(t)\}_{t\in\mathbb{R}}$, $i \in \mathbb{N}$ be a sequence of unipotent one-parameter subgroups of G and let $\{\nu_m : m \in \mathbb{N}\}$ be a sequence in $\mathcal{P}(\Gamma\backslash G)$ such that each ν_i is*

an ergodic $\{u_i(t)\}$-invariant measure. Suppose that $\nu_m \to \nu$ in $\mathcal{P}(\Gamma\backslash G)$ and let $x \in \operatorname{supp}(\nu)$. Then the following holds:

(1) $\operatorname{supp}(\nu) = x\Lambda(\nu)$ *where* $\Lambda(\nu) = \{g \in G : \nu g = \nu\}$.
(2) *Let $g_i' \to e$ be a sequence in G such that for every $i \in \mathbb{N}$, $xg_i' \in \operatorname{supp}(\nu_i)$ and the trajectory $\{xg_i' u_i(t)\}$ is uniformly distributed with respect to ν_i. Then there exists an $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,*

$$\operatorname{supp}(\nu_i) \subset \operatorname{supp}(\nu) g_i'.$$

(3) *ν is invariant and ergodic for the action of the subgroup generated by the set $\{g_i' u_i(t) g_i'^{-1} : i \geq i_0\}$.*

Proof of Proposition 2.2 Since $g_m^{-1}\Gamma g_m \cap H$ is an irreducible lattice in H, every non-compact simple factor of H acts ergodically on each $\Gamma\backslash\Gamma g_m H$ with respect to ν_m. There exists a unipotent one-parameter subgroup $U := \{u(t)\}$ in H_N not contained in any proper closed normal subgroup of H_N (cf. Lemma 2.3 [MS]). Then by Moore's ergodicity theorem (cf. Theorem 2.1 in [BM]), U acts ergodically with respect to each ν_m. Moreover by the Birkhoff ergodic theorem, the following subset R has the zero co-measure in H:

$$\{h \in H : \Gamma\backslash\Gamma g_m h u(t) \text{ is uniformly distributed in } \Gamma\backslash\Gamma g_m H \text{ w. r. t. } \nu_m \text{ for each } m \in \mathbb{N}\}.$$

Hence for any $h \in R$ and for any continuous bounded function f on X with compact support, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(g_m h u(t))\, dt = \int_X f\, d\nu_m.$$

If we set $L = \Lambda(\nu)$, we have that ν is supported on $\Gamma\backslash\Gamma c_0 L$ for some $c_0 \in \operatorname{supp}(\nu)$ by Theorem 2.3(1). There exist $\gamma_m \in \Gamma$ and $h_m \in R$ such that $\gamma_m g_m h_m \to c_0$ as $m \to \infty$. If we set

$$x_m := \gamma_m g_m h_m \quad \text{and} \quad g_m' := c_0^{-1} x_m,$$

then

$$\lim_{m\to\infty} g_m' = e$$

and $\Gamma\backslash\Gamma c_0 g_m' u(t) = \Gamma\backslash\Gamma g_m h_m u(t)$ is uniformly distributed with respect to ν_m. By Theorem 2.3(2), there exists an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$

$$\operatorname{supp}(\nu_m) \subset \operatorname{supp}(\nu) g_m', \quad \text{or equivalently} \quad \Gamma\backslash\Gamma g_m H \subset \Gamma\backslash\Gamma c_0 L g_m'.$$

Hence $\Gamma x_m H x_m^{-1} \subset \Gamma(c_0 L c_0^{-1})$. By the connectedness of H, we may assume that $c_0 L c_0^{-1}$ is connected and

$$\{x_m H x_m^{-1} : m \geq m_0\} \subset c_0 L c_0^{-1}.$$

By Theorem 2.3(3), the subgroup generated by the set $\{x_m H x_m^{-1} : m \geq m_0\}$ acts ergodically on $\Gamma c_0 L c_0^{-1}$. Hence $c_0 L c_0^{-1}$ is the smallest closed subgroup containing the subgroup generated by the set $\{x_m H x_m^{-1} : m \geq m_0\}$ such that the orbit $\Gamma c_0 L c_0^{-1}$ is closed. This proves (3). The second claim (2) follows from [MS, Proposition 2.1].

3. Proof of Theorem 1.2

For a G-space X and a subgroup M of G, $\mathcal{P}(X)^M$ denotes the space of M-invariant Borel probability measures on X. We recall the ergodic theoretic approach suggested in [BS]. Let $\Delta(G)$ be the diagonal embedding of G into $G\times G$, that is, $\Delta(G)=\{(g,g):g\in G\}$. For each $\nu\in\mathcal{P}(\Gamma\backslash G)^{\Gamma}$, the measure $\tilde\nu$ defined by

$$\tilde\nu(f):=\int_{\Gamma\backslash G}\int_{\Gamma\backslash G} f(g,hg)\,d\mu_G(g)d\nu(h)$$

for any bounded continuous function f on $\Gamma\backslash G\times\Gamma\backslash G$ is a $\Delta(G)$-invariant probability measure on $\Gamma\backslash G\times\Gamma\backslash G$. Moreover the map $\nu\mapsto\tilde\nu$ is a homeomorphism from $\mathcal{P}(\Gamma\backslash G)^{\Gamma}$ to $\mathcal{P}(\Gamma\backslash G\times\Gamma\backslash G)^{\Delta(G)}$. We make the following simple observation:

- For $a\in\mathrm{Comm}(\Gamma)$, if we set $\nu_a=\frac{1}{\deg(a)}\sum_{y\in\Gamma\backslash\Gamma a\Gamma}\delta_y$ where δ_y denotes the standard delta measure with support y, then $\tilde\nu_a$ is the (unique) $\Delta(G)$-invariant probability measure supported on $[(e,a)]\Delta(G)\subset(\Gamma\times\Gamma)\backslash(G\times G)$.
- The element $a\in G$ is contained in $\mathrm{Comm}(\Gamma)$ if and only if the orbit $[(e,a)]\Delta(G)$ is closed and supports a finite $\Delta(G)$-invariant measure.

Set $X=(\Gamma\times\Gamma)\backslash(G\times G)$ and consider its one point compactification $X\cup\{\infty\}$. By the fact that $\mathcal{P}(X\cup\{\infty\})$ is compact with respect to weak *-topology and the above observation, it suffices to show that, assuming the sequence $\{\tilde\nu_{a_i}\}$ weakly converging to $\tilde\nu$ in $\mathcal{P}(X\cup\{\infty\})$, the limit $\tilde\nu$ is $G\times G$-invariant and supported on X. Note that for each i, $(e,a_i)\Delta(G)(e,a_i^{-1})\cap(\Gamma\times\Gamma)$ is commensurable with $\Delta(G)\cap(\Gamma\times\Gamma)$. It follows that $(e,a_i)\Delta(G)(e,a_i^{-1})$ contains a Zariski dense subset contained in $\Delta(G)(\mathbb{Q})$. This implies that $(e,a_i)\Delta(G)(e,a_i^{-1})$ is a $\mathbb{Q}$-subgroup of $G\times G$ (cf. [Zi, Prop. 3.18]) and $\mathbb{Q}$-simple as well. Moreover, it is easy to see that the unique maximal connected normal subgroup of $(e,a_i)\Delta(G)(e,a_i^{-1})$ with no compact factors cannot be contained in any proper parabolic $\mathbb{Q}$-subgroup of $G\times G$ for each i. Therefore by Proposition 2.1, $\tilde\nu$ is supported on X. Also by applying Proposition 2.2, $\tilde\nu$ is either $G\times G$-invariant or $\Delta(G)$-invariant supported on $x\Delta(G)$ for some $x\in X$. Suppose that the latter case happens. Then Proposition 2.2 also says that there exist i and $y_j\in\Gamma\times\Gamma$ such that $y_j(e,a_j)\Delta(G)(e,a_j^{-1})y_j^{-1}=(e,a_i)\Delta(G)(e,a_i^{-1})$ for all $j\ge i$. That is, $(e,a_i^{-1})y_j(e,a_j)$ belongs to the normalizer of $\Delta(G)$ in $G\times G$ for all $j\ge i$. Since $\Delta(G)$ has a finite index in its normalizer in $G\times G$, we have $(e,a_i^{-1})y_j(e,a_j)\in\Delta(G)$ and hence $[(e,a_i)]\Delta(G)=[(e,a_j)]\Delta(G)$ for infinitely many j. This implies that $\Gamma a_i=\Gamma a_j$ and hence $\deg(a_j)$ is constant for infinitely many j. This is a contradiction, since $\deg(a_i)$ tends to ∞ as $i\to\infty$. Therefore $\tilde\nu$ is the $G\times G$-invariant probability measure supported on X, as desired.

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